

Exercice 1

$$a) \Omega = \frac{N!}{N_1! N_2!} = \frac{N!}{\left(\frac{N}{2}+n\right)! \left(\frac{N}{2}-n\right)!}$$

$$b) P(n) = \frac{\Omega(n)}{2^N} = \frac{1}{2^N} \cdot \frac{N!}{\left(\frac{N}{2}+n\right)! \left(\frac{N}{2}-n\right)!}$$

$$c) \ln P = \underbrace{\ln N! - \ln 2^N}_{\text{cte}} - \ln \left(\frac{N}{2}+n\right)! - \ln \left(\frac{N}{2}-n\right)!$$

$$\left\{ \begin{aligned} \ln \left(\frac{N}{2}+n\right)! &= \left(\frac{N}{2}+n\right) \ln \left(\frac{N}{2}+n\right) = \left(\frac{N}{2}+n\right) \left\{ \ln \frac{N}{2} + \ln \left(1 + \frac{2n}{N}\right) \right\} \\ \ln \left(\frac{N}{2}-n\right)! &= \left(\frac{N}{2}-n\right) \ln \left(\frac{N}{2}-n\right) = \left(\frac{N}{2}-n\right) \left\{ \ln \frac{N}{2} + \ln \left(1 - \frac{2n}{N}\right) \right\} \end{aligned} \right.$$

$$\left\{ \begin{aligned} \ln \left(\frac{N}{2}+n\right)! &= \left(\frac{N}{2}+n\right) \ln \frac{N}{2} + \left(\frac{N}{2}+n\right) \times \frac{2n}{N} \\ \ln \left(\frac{N}{2}-n\right)! &= \left(\frac{N}{2}-n\right) \ln \frac{N}{2} + \left(\frac{N}{2}-n\right) \left(-\frac{2n}{N}\right) \end{aligned} \right.$$

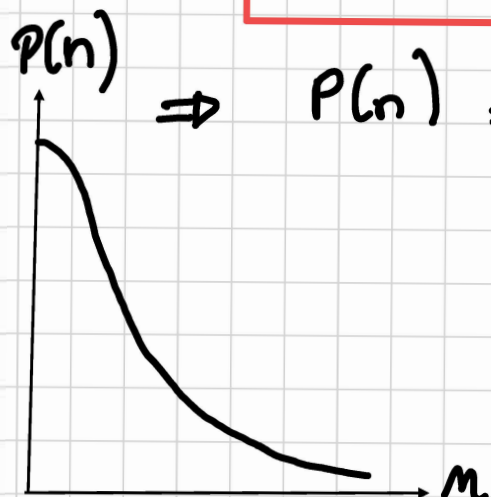
$$\Rightarrow \left\{ \begin{aligned} \ln \left(\frac{N}{2}+n\right)! &= \left(\frac{N}{2}+n\right) \ln \frac{N}{2} + \left(\frac{N}{2}+n\right) \times \frac{2n}{N} \\ \ln \left(\frac{N}{2}-n\right)! &= \left(\frac{N}{2}-n\right) \ln \frac{N}{2} + \left(\frac{N}{2}-n\right) \left(-\frac{2n}{N}\right) \end{aligned} \right.$$

$$\Rightarrow \ln \left(\frac{N}{2}+n\right)! + \ln \left(\frac{N}{2}-n\right)! = N \ln \frac{N}{2} + \frac{4n^2}{N}$$

$$\text{D'où : } \ln P = N \ln N - N \ln 2 - N \ln N + N \ln 2 - \frac{4n^2}{N}$$

$$\ln P = -\frac{4n^2}{N}$$

$$\Rightarrow P(n) = e^{-4n^2/N} \quad \text{avec } P(0) = 1$$



$$n_0 = 0$$

$$\Delta n \geq 0$$

$$\textcircled{2} \quad a) \quad \Omega = \frac{N!}{\left(\frac{N}{2}!\right)^2}$$

$$b) \quad \Delta S = S_{EF} - \underbrace{S_{EI}}_0 = k_B \ln \Omega = k_B \left\{ \ln N! - 2 \ln \frac{N}{2}! \right\}$$

$$= k_B \left\{ N \ln N - \cancel{\frac{N}{2}} \ln \frac{N}{2} \right\}$$

$$= k_B \left\{ N \ln N - N \ln N + N \ln 2 \right\}$$

$$\Delta S = k_B N \ln 2 = n R \ln 2 \quad \left(n = \frac{N}{N_A} \right)$$

$\textcircled{3}$ Approche classique Définir clairement le système

□ $(S) = \{ \text{les 2 récipients} \} \Rightarrow$ isolé thermiquement et mécaniquement

□ 1^{er} principe appliqué à S entre l'EI et l'EF :

$$\Delta U = \underbrace{\Delta U_{GP}}_{nR \Delta T} + \underbrace{\Delta U_{ide}}_0 = \underbrace{W}_0 + \underbrace{Q}_0 = 0 \Rightarrow \Delta T = 0$$

□ Variation d'entropie du GP :

$$\Delta S = \frac{nR}{\gamma-1} \ln \left(\frac{T_f V_f^{\gamma-1}}{T_i V_i^{\gamma-1}} \right) = nR \ln \left(\frac{V_f}{V_i} \right) \downarrow nR \ln 2$$

$V_f = 2V_i$ ↑ à prouver

$\Rightarrow$ On retrouve bien le même résultat.

Exercice 2

$\textcircled{1}$ □ $\Omega = \prod_i \Omega_i$ où $\Omega_i =$ nb de micro-états relatifs au GP _{i}

$$\square S = k_B \ln \Omega = k_B \ln \left(\prod_i \Omega_i \right) = \sum_i k_B \ln \Omega_i$$

$\Rightarrow S = \sum_i S_i$: Additivité des entropies.

$\textcircled{2}$ a) $(S) =$ ensemble de corps ds $V_1 \cup V_2 \Rightarrow$ système isolé

□ 1^{er} principe appliqué à (S):

$$\Delta U = W + Q = 0 = \Delta U_1 + \Delta U_2$$

$$\Rightarrow n_1 R (T_f - T_0) + n_2 R (T_f - T_0) = 0$$

$$\Rightarrow T_f = T_0$$

⚠ système hétérogène
 $\Rightarrow \Delta U = \sum_i \Delta U_i$

$$\square \begin{cases} P_0 V_1 = n_1 R T_0 \\ P_0 V_2 = n_2 R T_0 \end{cases} \Rightarrow P_0 (V_1 + V_2) = (n_1 + n_2) R T_0$$

□ EF: mélange de GP: $P_f (V_1 + V_2) = (n_1 + n_2) R T_0$

$$\Rightarrow P_f = P_0$$

b) $\Omega = \Omega_1 \times \Omega_2$

$$\begin{cases} \Omega_{1f} = K_1 V_{1f}^{N_1} U_{1f}^{\alpha_1 N_1} = k_1 U_{01}^{\alpha_1 N_1} (V_1 + V_2)^{N_1} \\ \Omega_{1i} = K_1 V_{1i}^{N_1} U_{1i}^{\alpha_1 N_1} = k_1 U_{01}^{\alpha_1 N_1} V_1^{N_1} \end{cases}$$

$$\begin{cases} \Omega_{2f} = K_2 V_{2f}^{N_2} U_{2f}^{\alpha_2 N_2} = k_2 U_{02}^{\alpha_2 N_2} (V_1 + V_2)^{N_2} \\ \Omega_{2i} = K_2 V_{2i}^{N_2} U_{2i}^{\alpha_2 N_2} = k_2 U_{02}^{\alpha_2 N_2} V_2^{N_2} \end{cases}$$

$$\begin{cases} \Omega_i = k \times V_1^{N_1} V_2^{N_2} \\ \Omega_f = k \times (V_1 + V_2)^{N_1 + N_2} \end{cases}$$

$$\begin{pmatrix} U_{01} = n_1 R T_0 = U_{1f} = U_{1i} \\ U_{02} = n_2 R T_0 = U_{2f} = U_{2i} \end{pmatrix}$$

on pose

$$K = k_1 k_2 U_{01}^{\alpha_1 N_1} U_{02}^{\alpha_2 N_2}$$

c) $\Delta S = k_B \ln \Omega_f - k_B \ln \Omega_i = k_B \ln \frac{(V_1 + V_2)^{N_1 + N_2}}{(V_1^{N_1} V_2^{N_2})}$

$$\Rightarrow \Delta S = k_B \ln \left(\frac{V_1 + V_2}{V_1} \right)^{N_1} + k_B \ln \left(\frac{V_1 + V_2}{V_2} \right)^{N_2}$$

$$\Rightarrow \Delta S = k_B N_1 \ln \left(\frac{V_1 + V_2}{V_1} \right) + k_B N_2 \ln \left(\frac{V_1 + V_2}{V_2} \right)$$

Approche classique:

$$\Delta S = \Delta S_1 + \Delta S_2 = \frac{n_1 R}{\gamma - 1} \ln \left(\frac{T_{1f} V_{1f}^{\gamma-1}}{T_{1i} V_{1i}^{\gamma-1}} \right) + \frac{n_2 R}{\gamma - 1} \ln \left(\frac{T_{2f} V_{2f}^{\gamma-1}}{T_{2i} V_{2i}^{\gamma-1}} \right)$$

$$\Delta S = n_1 R \ln \left(\frac{V_{1f}}{V_{1i}} \right) + n_2 R \ln \left(\frac{V_{2f}}{V_{2i}} \right) = n_1 R \ln \left(\frac{V_1 + V_2}{V_1} \right) + n_2 R \ln \left(\frac{V_1 + V_2}{V_2} \right)$$

(d) NON car dans ce cas $\Delta S = 0$: Les valeurs des Ω_1 et Ω_2 ne sont plus valables les particules étant indiscernables

Exercice 3

$$\textcircled{1} \quad \begin{cases} N_i = A e^{-E_i/k_B T} \\ \sum_i N_i = N \Rightarrow A = \frac{N}{\sum_i e^{-E_i/k_B T}} \end{cases} \quad N_i = N \frac{e^{-E_i/k_B T}}{\sum_i e^{-E_i/k_B T}}$$

$$x \quad P_i = \frac{N_i}{N} = \frac{e^{-E_i/k_B T}}{Z}$$

$$\textcircled{2} \quad U = \sum_i E_i N_i = N \sum_i E_i \frac{e^{-E_i/k_B T}}{Z}$$

$$\text{OR: } Z = \sum_i e^{-E_i/k_B T} \Rightarrow \frac{dZ}{dT} = + \frac{1}{k_B T^2} \sum_i E_i e^{-E_i/k_B T}$$

$$\text{D'où: } U = N k_B T^2 \frac{1}{Z} \frac{dZ}{dT} \Rightarrow U = N k_B T^2 \frac{d}{dT} (\ln Z)$$

$$\textcircled{3} \textcircled{1} \quad S = -k_B \sum_i P_i' \ln P_i' \quad (\text{HP}) \quad \text{ou} \quad P_i' = \prod_i P_i = \prod_i \frac{1}{Z_i} e^{-E_i/k_B T}$$

sachant que: $\forall i \quad Z_i = Z$ et en notant $E = \sum_i E_i$, on obtient:

$$P_i' = \frac{1}{Z^N} e^{-E/k_B T} = \frac{1}{Z'} e^{-E/k_B T} \quad \text{ou} \quad Z' = Z^N \quad \triangle P_i' \neq P_i$$

D'où:

$$\begin{aligned} S &= -k_B \sum_i \frac{1}{Z'} e^{-E/k_B T} \left(-\frac{E}{k_B T} - \ln Z' \right) \\ &= \frac{k_B}{Z'} \sum_i \left(\frac{E}{k_B T} e^{-E/k_B T} + \ln Z' e^{-E/k_B T} \right) \end{aligned}$$

$$= \frac{k_B}{Z'} \left\{ \frac{1}{k_B T} \left(k_B T^2 \frac{dZ'}{dT} \right) + \ln Z' \sum_i e^{-E/k_B T} \right\}$$

$$= \frac{k_B}{Z'} \left\{ T \frac{dZ'}{dT} + \ln Z' \times Z' \right\} = k_B \left\{ \frac{T}{Z'} \frac{dZ'}{dT} + \ln Z' \right\}$$

$$= k_B \frac{d}{dT} (T \ln Z') = k_B \frac{d}{dT} (T \ln Z^N) \Rightarrow S = N k_B \frac{d}{dT} (T \ln Z)$$

Méthode 1

2

$$S = -k_B \ln \Omega.$$

$$\Omega = \frac{N!}{\prod_i N_i!}$$

$$\ln \Omega = \ln N! - \ln \left(\prod_i N_i! \right) = \ln N! - \sum_i \ln N_i!$$

$$= N \ln N - \sum_i N_i \ln N_i$$

$$= N \ln N - \sum_i \frac{N}{Z} e^{-E_i/k_B T} \left(\ln N - \ln Z - \frac{E_i}{k_B T} \right)$$

$$\Rightarrow \ln \Omega = N \ln N - \frac{N}{Z} \left\{ \ln N \times Z - \ln Z \times Z - \frac{1}{k_B T} (k_B T^2 \frac{dZ}{dT}) \right\}$$

$$= N \left\{ \ln Z + \frac{T}{Z} \frac{dZ}{dT} \right\}$$

$$= N \frac{d}{dT} (T \ln Z)$$

Donc :

$$S = k_B N \frac{d}{dT} (T \ln Z)$$

$$(4) \times P_i = \frac{e^{-E_i/k_B T}}{\sum_q e^{-E_q/k_B T}} = \left\{ \sum_q \underbrace{e^{(E_i - E_q)/k_B T}}_{\alpha_i} \right\}^{-1}$$

$$\times \text{Cas où } 0 < i < N \begin{cases} \text{si } E_i = E_q \text{ alors } \alpha_i = 1 \\ \text{si } E_i > E_q \text{ alors } \alpha_i \xrightarrow{T \rightarrow 0} \infty \\ \text{si } E_i < E_q \text{ alors } \alpha_i \xrightarrow{T \rightarrow 0} 0 \end{cases}$$

 $\Rightarrow P_i$ tend vers 0

$$\times \text{Cas où } i = N \begin{cases} \text{si } E_i > E_q \text{ alors } \alpha_i \rightarrow \infty \\ \text{si } E_i = E_q = E_N \text{ alors } \alpha_i = 1 \end{cases}$$

 $\Rightarrow P_N$ tend vers 0

$$\times \text{Cas où } i = 0 \begin{cases} \text{si } E_i < E_q \text{ alors } \alpha_i \rightarrow 0 \\ \text{si } E_i = E_q = E_0 \text{ alors } \alpha_i = 1 \end{cases}$$

 $\Rightarrow P_0$ tend vers 1

$\Rightarrow$ A $T=0$, le seul état occupé est celui de plus petite énergie.

$$\Rightarrow \Omega = 1$$

$$\Rightarrow S = 0$$

Exercice 4

$$\begin{aligned} \textcircled{1} \quad \langle m \rangle &= m \frac{N^+}{N} - m \frac{N^-}{N} = \frac{m}{N} (N^+ - (N - N^+)) \\ &= \frac{m}{N} (2N^+ - N) = \frac{m}{N} ((1+z)N - N) \Rightarrow \langle m \rangle = mz \end{aligned}$$

$$\textcircled{2} \quad S = k_B \ln \Omega$$

$$\Omega = \frac{N!}{N_+! N_-!} = \frac{N!}{\left(\left(1+\frac{z}{2}\right)\frac{N}{2}\right)! \times \left(\left(1-\frac{z}{2}\right)\frac{N}{2}\right)!}$$

$$\ln \Omega = N \ln N - \left(1+\frac{z}{2}\right)\frac{N}{2} \ln \left(1+\frac{z}{2}\right)\frac{N}{2} - \left(1-\frac{z}{2}\right)\frac{N}{2} \ln \left(1-\frac{z}{2}\right)\frac{N}{2}$$

$$\text{Avec: } \left(1+\frac{z}{2}\right)\frac{N}{2} \ln \left(1+\frac{z}{2}\right)\frac{N}{2} = \left(\frac{N}{2} + \frac{Nz}{2}\right) (\ln(1+z) + \ln N - \ln 2)$$

$$\left(1-\frac{z}{2}\right)\frac{N}{2} \ln \left(1-\frac{z}{2}\right)\frac{N}{2} = \left(\frac{N}{2} - \frac{Nz}{2}\right) (\ln(1-z) + \ln N - \ln 2)$$

La somme vaut dc :

$$\begin{aligned} &\frac{N}{2} (\ln(1+z) + \ln(1-z)) + N \ln N - N \ln 2 \\ &+ \frac{N}{2} z (\ln(1+z) - \ln(1-z)) \end{aligned}$$

$$\text{D'où: } \ln \Omega = N \ln 2 - \frac{N}{2} \ln(1-z^2) - \frac{N}{2} z \ln \left(\frac{1+z}{1-z}\right)$$

$$\text{Donc: } S = k_B N \ln 2 - \frac{N}{2} k_B \left\{ \ln(1-z^2) + z \ln \left(\frac{1+z}{1-z}\right) \right\}$$

$$\textcircled{3} \quad \begin{cases} \langle e^+ \rangle = \frac{N^+}{N} (-J) + \frac{N^-}{N} (J) \\ \langle e^- \rangle = \frac{N^+}{N} (J) + \frac{N^-}{N} (-J) \end{cases}$$

$$\begin{cases} \langle e^+ \rangle = \left(-\left(\frac{1+z}{2}\right) + \left(\frac{1-z}{2}\right) \right) \mathcal{I} \\ \langle e^- \rangle = \left(+\left(\frac{1+z}{2}\right) - \left(\frac{1-z}{2}\right) \right) \mathcal{I} \end{cases} \Rightarrow \begin{cases} \langle e^+ \rangle = -\mathcal{I}z \\ \langle e^- \rangle = +\mathcal{I}z \end{cases}$$

$$\text{Alors } U = N_+ \langle e^+ \rangle + N_- \langle e^- \rangle$$

$$= \frac{N}{2} \left\{ (1+z)(-\mathcal{I}z) + (1-z)(+\mathcal{I}z) \right\}$$

$$\Rightarrow U = -\mathcal{I}Nz^2$$

$$(4) \quad U = -\mathcal{I}Nz^2 \Rightarrow dU = -2\mathcal{I}Nz dz$$

$$S = Nk_B \ln 2 - \frac{N}{2} \left\{ \ln(1-z^2) + z \ln\left(\frac{1+z}{1-z}\right) \right\}$$

$$\begin{aligned} \Rightarrow dS &= -\frac{Nk_B}{2} \left\{ \frac{-2z dz}{1-z^2} + dz \ln\left(\frac{1+z}{1-z}\right) + z \left(\frac{dz}{1+z} + \frac{dz}{1-z} \right) \right\} \\ &= -\frac{Nk_B}{2} \ln\left(\frac{1+z}{1-z}\right) dz \end{aligned}$$

$$dS = -Nk_B \text{Argh}(z) dz$$

$$\text{Argh}(z) = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right)$$

$$\text{Donc : } T = \frac{dU}{dS} = \frac{2\mathcal{I}z}{k_B \text{Argh}(z)}$$

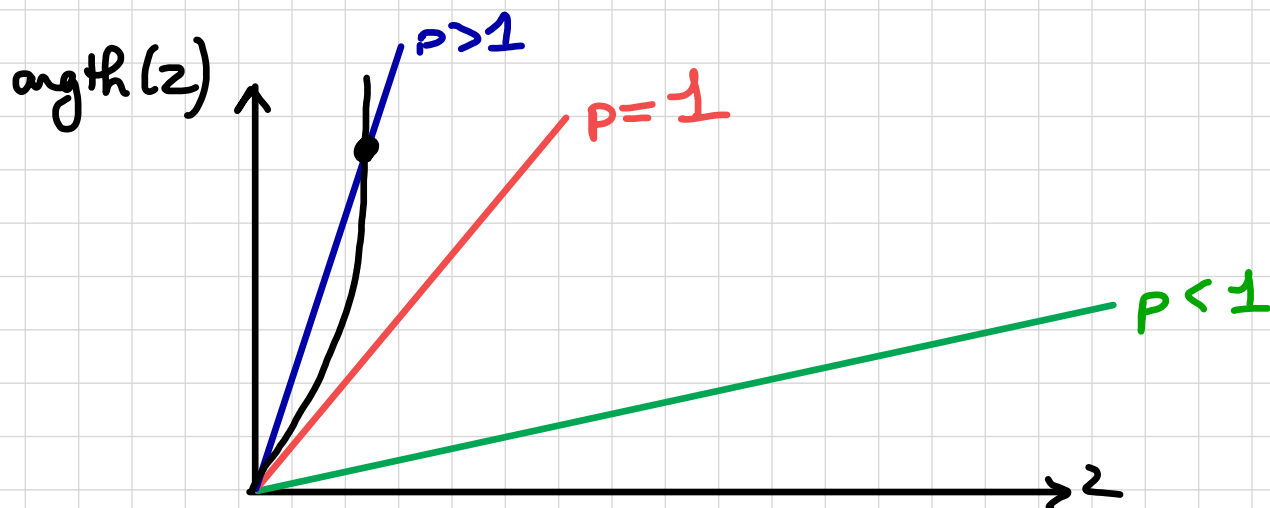
on a donc :

$$\text{Argh}(z) = \underbrace{\left(\frac{2\mathcal{I}}{k_B T} \right)}_p z$$

Méthode graphique:

Intersection des 2 courbes

(on note $T_c / p = 1$)



$\Rightarrow$ pt d'intersection ($p, z \neq 0$) uniquement $p > 1 \Rightarrow$

$$T < T_c = \frac{2\mathcal{I}}{k_B}$$

Exercice 5

$$\textcircled{1} \quad L(n) = L n + (L-a)(N-n) = N(L-a) + na$$

$$\textcircled{2} \quad E(n) = n e_1 + (N-n) e_2 = N e_2 - n \Delta e$$

$$\textcircled{3} \quad \Omega = \frac{N!}{n!(N-n)!}$$

$$\textcircled{4} \quad \ln \Omega = N \ln N - n \ln n - (N-n) \ln (N-n)$$

$$S = k_B \ln \Omega = k_B \left\{ N \ln \left(\frac{N}{N-n} \right) - n \ln \left(\frac{n}{N-n} \right) \right\}$$

$$\textcircled{5} \quad F(n) = E(n) - TS(n)$$

$$= N e_2 - n \Delta e - k_B T \left\{ N \ln \left(\frac{N}{N-n} \right) - n \ln \left(\frac{n}{N-n} \right) \right\}$$

$$\frac{\partial F}{\partial n} = -\Delta e - k_B T \left\{ \cancel{\frac{N}{N-n}} - \ln \left(\frac{n}{N-n} \right) - \cancel{1} - \cancel{\frac{n}{N-n}} \right\}$$

$$\frac{\partial F}{\partial n} = -\Delta e + k_B T \ln \left(\frac{n}{N-n} \right) = 0 \quad \text{or } n = \tilde{n}$$

$$\Rightarrow \tilde{n} = \frac{N}{1 + e^{-\Delta e / k_B T}}$$

$$\Rightarrow L(\tilde{n}) = N(L-a) + \tilde{n} a$$

$$= N(L-a) + \frac{Na}{1 + e^{-\Delta e / k_B T}}$$

$$= N \left\{ L + a \left(-1 + \frac{1}{1 + e^{-\Delta e / k_B T}} \right) \right\}$$

$$L(\tilde{n}) = N \left\{ L - \frac{a}{1 + e^{+\Delta e / k_B T}} \right\}$$

Car $T \uparrow \Rightarrow L(\tilde{n}) \downarrow \Rightarrow$ La laine rétrécit à l'eau chaude